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Hecke algebra representations within Clifford geometric algebras of multivectors

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Abstract. We introduce Clifford geometric algebras of multivectors which exhibit a bilinear form which is not necessarily symmetric. Looking at a subset of bi-vectors in $C\ell(\mathbb{K}^{2n}, B)$, we prove that these elements provide a representation of the Hecke algebra $H_{\mathbb{K}}(n+1, q)$ if the bilinear form B is chosen appropriately. This shows that q-quantization can be generated by Clifford multivector objects which usually describe composite entities. This contrasts current approaches which give deformed versions of Clifford algebras by deforming the one-vector variables. Our example shows that it is not evident, from a mathematical point of view, that q-deformation is in any sense more elementary than the undeformed structure.

1. Introduction

Recent developments in theoretical physics employ the so-called *non-commutative geometry* [1] or in a more special case *q-deformed geometry* [2–4]. The underlying structure is either C^*-theory, which also incorporates topological and convergence aspects, or Hopf algebras, which model the algebraic aspects of a theory [5, 6]. It is convenient to speak of q-symmetry since the spaces on which q-symmetry acts tend to be *braided*. It is thus convenient to study braided monoidal categories, see, for example, [2, 7, 8]. The main idea is to introduce a braided tensor product algebra structure

$$(a \otimes b)(c \otimes d) = a\Psi(b \otimes c)d \tag{1}$$

where Ψ is a braiding. If Ψ is trivial or minus the flip operator $\Psi(a \otimes b) = -b \otimes a$, one deals with the ordinary tensor product (bosons) or a $\mathbb{Z}_2$-graded version of it (fermions). A general braiding thus leads to general or braid statistics. The central relations obeyed by braid groups are the Artin braid relations [9]

$$\begin{aligned} &t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1} \\ &t_i t_k = t_k t_i \qquad |i-k| \geqslant 2. \end{aligned} \tag{2}$$

The first is actually equivalent to the so-called quantum Yang–Baxter equation, which is a special case of the Yang–Baxter equation [10] (in standard notation)

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(v) \tag{3}$$

with the spectral parameters set to $v = u + v = u$.

There has been great progress in the theory of (quantum) statistical mechanics, which originated in the development of the inverse scattering method [11] and the star triangle

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relation [12], both methods having their roots in braided symmetries, see, for example, [13]. There are now many models which are solvable by these methods: the Ising [14] and N-state Potts models [15, 16] and the Vertex [17] and IRF models [18] are prominent examples. A further example might be given by the (fractional) quantum Hall effect [19]. Furthermore, the unexpected connection between link invariants and type II subfactors of von Neumann algebras unveiled by Jones has pushed low-dimensional topology far ahead, see [20, 21]. There is even a connection of the Jones polynomial to quantum field theory [22].

A further branch of applications arises from the common belief that q-symmetry is more general than the usual bosonic or fermionic symmetries and is thus more fundamental, see, for example, [2, 23, 24]. The natural thing to do is therefore to provide q-deformed versions of physical relevant groups, for example the Poincaré group [25]. There is a strong belief that the fundamental constant $\hbar$ is involved in this construction and that spacetime should behave as if it were 'q-symmetric' at small scales.

The above-mentioned situations, when q-symmetry leads to explicit results, share the feature of being effective or composite models. There is no recent *evidence* that q-symmetry has to be used in fundamental interactions. Moreover, it might be expected that a q-deformed Poincaré group has an underlying structure which generates this symmetry.

From a mathematical viewpoint, there is no harm in q-deforming all structures which can be so changed. However, a physical application requires an interpretation which seems currently not to be obvious, but relies on rather abstract developments such as quantum planes and q-deformed or non-commutative geometry.

We are thus in a perplexing situation, because in general q-deformation can be applied to nearly every mathematical structure which is currently used in physics. However, we do not know in which cases this might be reasonable. To be able to answer this question, it is a valuable advantage to embed the mathematical structure which lays at the heart of q-deformation, the Hecke algebra, into a larger framework. From this outstanding point of view it might be possible to decide if q-symmetry has to be applied to, for example, gravitation or not.

A very interesting approach to q-symmetry using spinors, and thereby also with the help of Clifford algebras, can be found in [7]. This approach, however, takes q-symmetry as an elementary property. In the same spirit, the Clifford algebra of a Hecke braid was constructed in [26].

We will provide a theorem which shows that Hecke algebra representations can be obtained within certain Clifford algebras, being faithful in low dimensions. These representations are generated by bi-vector elements and thus by objects which are composed. Furthermore, since the interpretation of Clifford algebraic expressions is well known, we come to the conclusion that q-symmetry is tightly interwoven with *composite* structures, as was suggested already in [27]. This relation is seen from the fact that q-symmetry is obtained in this approach as a multivector symmetry. It is this relation that makes us so suspicious of a q-deformation of spacetime as long as one does not have a *microscopic description* of these entities. Hopefully our approach will open the possibility of clarifying this situation.

2. Clifford geometric algebra of multivectors

There are many possibilities of introducing Clifford algebras, each of them emphasizing a different point of view. In our case, it is of utmost importance to have the Clifford algebra built over a graded linear space. This grading is obtained from the space underlying a Grassmann algebra. The Clifford algebra is then related to the endomorphism algebra of this Grassmann algebra. This construction, the Chevalley deformation [28], was originally invented to be able to treat Clifford algebras over fields of $char = 2$, see the appendix of [29] by Lounesto and

[36]. However, we use this construction in an entirely different context. With the help of the construction of Riesz [29], one is able to reconstruct the multivector structure and thereby a correspondence between the linear spaces underlying the Clifford algebra and the Grassmann algebra in use. This reconstruction depends on an automorphism J, which is arbitrary, see [30]. In fact this is just the reversed direction of our construction given below following Chevalley.

Let $T(V)$ be the tensor algebra built over the $\mathbb{K}$-linear space V. The field $\mathbb{K}$ will be either $\mathbb{R}$ or $\mathbb{C}$. With $V^0 \simeq \mathbb{K}$ we have

$$T(V) = \mathbb{K} \oplus V \oplus V \otimes_{\mathbb{K}} V \oplus \cdots. \tag{4}$$

The tensor algebra is associative and unital. In $T(V)$ one has bilateral or two-sided ideals, which can be used to construct new algebras by factorization. As an example, we define the Grassmann algebra in this way.

Definition 1. *The Grassmann algebra $\bigwedge(V)$ is the factor algebra of the tensor algebra wrt the bilateral ideal*

$$I_{\mathrm{Gr}} = \{y \mid y = a \otimes x \otimes x \otimes b \; a, b \in T(V), x \in V\}$$

$$\bigwedge(V) = \pi(T(V)) = \frac{T(V)}{I_{\mathrm{Gr}}} = \mathbb{K} \oplus V \oplus V \wedge V \oplus \cdots. \tag{5}$$

The canonical projection $\pi : T(V) \mapsto \bigwedge(V)$ maps the tensor product $\otimes$ onto the exterior or wedge product denoted by $\wedge$.

One may note that the factorization preserves the grading naturally inherited by the tensor algebra, since the ideal I_{Gr} is homogeneous. Defining the homogeneous parts of $T(V)$ and $\bigwedge(V)$ by $T^k(V) = V \otimes \cdots \otimes V$ and $\bigwedge^k(V) = V \wedge \cdots \wedge V$, k-factors, we obtain $\pi(T^k(V)) = \bigwedge^k(V)$.

Proceeding to Clifford algebras requires a further structure, the quadratic form.

Definition 2. *The map $Q : V \mapsto \mathbb{K}$, satisfying ($\alpha \in \mathbb{K}, x, y \in V$)*

(i) $$Q(\alpha x) = \alpha^2 Q(x) \tag{6a}$$

(ii) $$B_p(x, y) = \tfrac{1}{2}(Q(x + y) - Q(x) - Q(y)) \tag{6b}$$

where $B_p(x, y)$ is a symmetric bilinear form, is called a quadratic form.

It is tempting to introduce an ideal $I_{\mathrm{C}\ell}$

$$I_{\mathrm{C}\ell} = \{y \mid y = a \otimes (x \otimes x - Q(x)\mathbb{1}) \otimes b, a, b \in T(V), x \in V\} \tag{7}$$

to obtain the Clifford algebra by a factorization procedure. However, since we are interested in arbitrary bilinear forms underlying a Clifford algebra, we will take another approach, which is much more reasonable for such a structure. Furthermore, the Clifford algebra *does not* have an intrinsic multivector structure, but is *only* $\mathbb{Z}_2$ graded, since the ideal $I_{\mathrm{C}\ell}$ is inhomogeneous.

Let V^* be the space of linear forms on V, i.e. $V^* \simeq \mathrm{lin}[V, \mathbb{K}]$. Elements $\omega \in V^*$ act on elements $x \in V$, but there is *no natural* identification between V and V^*. However, we can find a set of x_i which spans V and dual elements ω_k acting on the x_i in a *canonical* way

$$\omega_k(x_i) = \delta_{ki}. \tag{8}$$

This allows the introduction of a map $* : V \mapsto V^*, x_i^* = \omega_i$, which may be called the Euclidean dual isomorphism [31]. The two spaces (V^*, V) connected by this duality constitute a pairing $\langle \cdot \mid \cdot \rangle : V^* \times V \mapsto \mathbb{K}$. V^* is isomorphic to V in finite dimensions, so it is natural to build a Grassmann algebra $\bigwedge(V^*)$ over it. This is the algebra of Grassmann multiforms.

It is therefore natural to extend the pairing of the grade-one space and its dual to the whole algebras $\bigwedge(V)$ and $\bigwedge(V^*)$, as can be seen by its frequent occurrence in the literature [30, 32–36] and others. This can be seen in the following.

Definition 3. *Let $\tau, \eta \in \bigwedge(V^*)$, $\omega \in V^*$, $u, v \in \bigwedge(V)$ and $x \in V$, then we can define a canonical action of $\bigwedge(V^*)$ on $\bigwedge(V)$ requiring*

$$\text{(i)} \qquad \omega(x) = \langle \omega \mid x \rangle \tag{9a}$$
$$\text{(ii)} \qquad \omega(u \wedge v) = w(u) \wedge v + \hat{u} \wedge \omega(v) \tag{9b}$$
$$\text{(iii)} \qquad (\tau \wedge \eta)(u) = \tau(\eta(u)) \tag{9c}$$

where $\hat{u}$ is the main involution $\hat{V} = -V$ extended to $\bigwedge(V)$.

In definition 3 we have in fact given an isomorphism between the Grassmann algebra of multiforms $\bigwedge(V^*)$ and the dual Grassmann algebra $[\bigwedge(V)]^*$. This can be made much clearer by writing

$$y \,\lrcorner\, x = \omega_y(x) = \langle \omega_y \mid x \rangle = B(y, x) \tag{10}$$

where we have used the canonical identification of V and V^* via the pairing. One should be very careful in the distinction of $\bigwedge(V^*)$ and $[\bigwedge(V)]^*$, since they are isomorphic but not equivalent. Furthermore, we emphasize that in writing $y \,\lrcorner$ we make explicit use of a *special* dual isomorphism encoded in the contraction

$$\begin{aligned} . \lrcorner : V &\mapsto V^* \\ y &\to y \,\lrcorner = \omega_y. \end{aligned} \tag{11}$$

Since there is no natural, that is mathematically motivated or even better functorial, relation between V and V^*, we are called to seek for *physically motivated reasons* to select a pairing. This freedom will enable us in section 3 to give a proof of our main theorem.

Theorem 4. *Let (V, Q) be a pair of a $\mathbb{K}$-linear space V and a quadratic form Q as defined in definition 2. There exists an injection γ, called a Clifford map from V into the associative unital algebra $C\ell(V, Q)$, which satisfies*

$$\gamma_x \gamma_x = Q(x)\mathbb{1}. \tag{12}$$

Definition 5. *The (smallest) algebra $C\ell(V, Q)$ generated by $\mathbb{1}$ and γ_{x_i}, $\{x_i\}$ span V, is called (the) Clifford algebra of Q over V.*

By polarization of the relation (12) we get the usual commutation relations, $x, y \in V$,

$$\gamma_x \gamma_y + \gamma_y \gamma_x = 2B_p(x, y)\mathbb{1} \tag{13}$$

where $B_p(x, y)$ is the symmetric polar form of Q as defined in (6).

Remarks. (i) We could have obtained this result directly by factorization of the tensor algebra with the ideal (7). (ii) There exist Clifford algebras which are universal, in this case it is convenient to speak from *the* Clifford algebra over (V, Q). (iii) If $V \simeq \mathbb{K}^n \simeq \mathbb{C}^n$ or $\mathbb{R}^n$, we denote $CL(V, Q)$ also by $CL(\mathbb{C}^n) \simeq C\ell_n$ or $C\ell(\mathbb{R}_{p,q})$ where the pair p, q enumerate the number of positive and negative eigenvalues of Q. We can also give the dimension n and signature $s = p - q$ to classify all quadratic forms over $\mathbb{R}$. In the case of the complex field, one remains with the dimension as can be seen, for example, from the Weyl unitary trick, letting $x_i \to ix_i$ which flips the sign. We do not use sesquilinear forms here, which could nevertheless be included.

We will now use Chevalley deformations to construct the Clifford algebra of multivectors. The main idea is that one can decompose the Clifford map as

$$\gamma_x = x \,\lrcorner + x \wedge . \tag{14}$$

There is thus a natural action of γ_x on $\bigwedge(V)$.

Theorem 6 (Chevalley). *Let $\bigwedge(V)$ be the Grassmann algebra over V and $\gamma : V \mapsto \mathrm{End}(\bigwedge(V))$ be defined as in (14), then γ is a Clifford map.*

We have shown that $C\ell$ is a subalgebra of the endomorphism algebra of $\bigwedge(V)$,

$$C\ell \subseteq \bigwedge(V). \tag{15}$$

It is possible to interpret $x \lrcorner$ and $x\wedge$ as annihilating and creation *operators* (on the Grassmann algebra) [37].

With the help of relations (9) we can then lift this Clifford map to multivector actions. *No symmetry requirement has to be made on the contraction.* This leads to the following definition.

Definition 7 (Clifford algebra of multivectors). *Let $B : V \times V \mapsto \mathbb{K}$ be an arbitrary bilinear form. The Clifford algebra $C\ell(V, B)$ obtained from lifting the Clifford map*

$$\gamma_x = x \lrcorner + x\wedge = \langle x \mid \cdot\rangle + x\wedge = B(x, \cdot) + x\wedge \tag{16}$$

to $\mathrm{End}(\bigwedge(V))$ *using relations (9) is called Clifford algebra of multivectors.*

Note that $B(x, \cdot) = \omega_x$ is a map from $V \to V^*$ and *incorporates* a dual isomorphism. It is clear from the construction that $C\ell(V, B)$ has a multivector structure or say a $\mathbb{Z}_n$-grading inherited from the Grassmann algebra $\bigwedge(V)$.

B admits a decomposition into symmetric and antisymmetric parts $B = G + F$. The symmetric part $G = B_p$ corresponds to a quadratic form Q, see definition 2.

Theorem 8. *The Clifford algebra $C\ell(V, Q) \simeq C\ell(V, G)$ is isomorphic as Clifford algebra to $C\ell(V, B)$, if B admits a decomposition $B = G + F$, $G^{\mathrm{T}} = G$, $F^{\mathrm{T}} = -F$.*

A proof can be found for low dimensions in [35] and in general in [8]. However, this result is implicitly known to physicists, see [38, 39]. In fact, this is the old Wick rule of quantum field theory. We will insist on the $\mathbb{Z}_n$-grading and therefore carefully distinguish Clifford algebras of multivectors with a common quadratic form Q but different contractions B. Only this generalization makes it possible to find a Hecke algebra representation in Clifford algebras.

We give some further notation. Let $\{j_i\}$ be a set of elements spanning $V \simeq \langle j_1, \dots, j_n\rangle$ and $\{\partial_k\}$ be a set of dual elements. Building the Grassmann algebras $\bigwedge(V)$ and $\bigwedge(V^*)$ and defining the action of the forms via (9), one obtains the relations

$$\text{(i)} \qquad j_i \wedge j_i = 0 = \partial_i \wedge \partial_i \tag{17a}$$

$$\text{(ii)} \qquad \partial_i j_k + j_k \partial_i = B_{ik} + B_{ki} = 2G_{ik}. \tag{17b}$$

The space $\boldsymbol{V} = V \oplus V^*$ is thus spanned by (note the reversed order of indices for the ∂ elements)

$$\{e_1, \dots, e_{2n}\} = \{j_1, \dots, j_n, \partial_n, \dots, \partial_1\}. \tag{18}$$

To have a simple notation, we introduce barred indices $i \in \{1, \dots, n\}$

$$e_{\bar{i}} = e_{2n+1-i} \tag{19}$$

or equivalently

$$e_i = j_i \qquad e_{\bar{i}} = e_{2n+1-i} = \partial_i. \tag{20}$$

The contraction on $\boldsymbol{V}$ in $C\ell(\boldsymbol{V}, B)$ is then written as

$$[B(e_i, e_j)] = [B_{ij}] = \begin{vmatrix} B^{jj}_{uv} & B^{j\partial}_{st} \\ B^{\partial j}_{rq} & B^{\partial\partial}_{xy} \end{vmatrix} \Leftrightarrow \begin{vmatrix} [M_{rs}]^{jj} & [B^1_{t\bar{u}}]^{j\partial} \\ [B^2_{\bar{v}w}]^{\partial j} & [N_{\bar{x}\,\bar{y}}]^{\partial\partial} \end{vmatrix} \tag{21}$$

where the superscripts indicate the type of base element. Indices of blocks run in $\{1\dots n\}$. Note that the matrices B^1, B^2 and N are not directly submatrices of B because of our barred index notation. Introducing an $n\times n$ matrix $J=\delta_{i,n+1-k}$ we can identify them as

$$B^{j\partial}=B^1J \qquad B^{\partial j}=JB^2 \qquad B^{\partial\partial}=JNJ. \tag{22}$$

We could handle the $2n$-dimensional complex case as $C\ell(\mathbb{R}^{2n+1},B)$, but we will restrict ourselves to the even-dimensional case and look at $C\ell(\mathbb{C}^{2n})\simeq\mathbb{C}\otimes C\ell(\mathbb{R}^{2n})$ as a complexification.

3. Hecke algebra representations over Clifford bi-vector generators

3.1. The definition and main theorem on the algebra morphism

Definition 9. *The Hecke algebra $H_{\mathbb{K}}(n+1,q)$ has the following presentation*

(i) $\quad t_i^2=(1-q)t_i+q\mathbb{1}$ $\quad$ *Hecke condition* $\quad$ (23a)

(ii) $\quad t_it_k=t_kt_i,\ |i-k|\geqslant 2$ $\quad$ *commutator* $\quad$ (23b)

(iii) $\quad t_it_{i+1}t_i=t_{i+1}t_it_{i+1}$ $\quad$ *Artin braid relation [9]* $\quad$ (23c)

with generators $\mathbb{1}$, b_i, $i\in\{1,\dots,n\}$, see [40].

Our goal is to find an algebra homomorphism of the t_i generators as bi-vectors in an appropriate Clifford algebra $C\ell(\mathbb{R}^{2n},B)$ or $C\ell(\mathbb{C}^{2n},B)$. We can formulate our result in the following.

Theorem 10. *A representation ρ of the Hecke algebra $H_{\mathbb{K}}(n+1,q)$ of definition 9 can be found in the Clifford algebra of multivectors $C\ell(\mathbb{K}^{2n},B)$ of definition 7 with the following identifications:*

(i) $\quad \rho(t_i)=b_i:=e_i\wedge e_{\bar{i}}=e_i\wedge e_{2n+1-i}\equiv j_i\wedge\partial_i \qquad i\in\{1,\dots,n\}$ $\quad$ (24a)

(ii) $\quad B:=[B_{ij}]=\begin{vmatrix}[B_{rs}]^{jj} & [B_{tu}]^{j\partial}\\ [B_{vw}]^{\partial j} & [B_{xy}]^{\partial\partial}\end{vmatrix}$ $\quad$ (24b)

where the submatrices of B satisfy the conditions

$$\begin{aligned}
B^{jj}&\equiv M_{rs}=\tfrac{1}{2}(M_{rs}-M_{sr})\\
JB^{\partial\partial}J&\equiv N_{\bar{x}\bar{y}}=\tfrac{1}{2}(N_{\bar{x}\bar{y}}-N_{\bar{y}\bar{x}})\\
B^{j\partial}J&\equiv[B^1_{t\bar{u}}]=[B_{t\bar{u}}+(q-B_{t\bar{u}})\delta_{t,\bar{u}}]\\
JB^{\partial j}&\equiv[B^2_{\bar{v}w}]=[-B_{w\bar{v}}+(1+q)\delta_{w,\bar{v}}+q\delta_{w+1,\bar{v}}+\delta_{w,\bar{v}+1}].
\end{aligned}\tag{25}$$

Proof. We determine the constraints on the bilinear form B by (23).

(i) We try to identify the bi-vector elements b_i from (24) with Hecke generators t_i from (23). Since we insist on the multivector structure inherited from the Grassmann multivectors underlying the Clifford multivectors, we have to fulfil in any case the condition

$$e_ie_i=B_{ii}+e_i\wedge e_i=0 \qquad (B_{\bar{i}\,\bar{i}}=0). \tag{26}$$

The Hecke relation (23a) leads with

$$b_i=j_i\wedge\partial_i=j_i\partial_i-B_{i\bar{i}} \tag{27}$$

to

$$\begin{aligned} b_i^2 &= (j_i \wedge \partial_i)^2 \\ &= (j_i \partial_i - B_{i\bar{i}}) j_i \wedge \partial_i \\ &= j_i [B_{i\bar{i}} \partial_i - j_i \partial_i^2] - B_{i\bar{i}} j_i \wedge \partial_i \\ &= B_{\bar{i}\,i} B_{i\bar{i}} - (B_{\bar{i}i} - B_{i\bar{i}}) j_i \wedge \partial_i \\ &= B_{\bar{i}\,i} B_{i\bar{i}} - (B_{\bar{i}i} - B_{i\bar{i}}) b_i \\ &\stackrel{!}{=} (1-q) b_i + q. \end{aligned} \tag{28}$$

We obtain as solutions

$$\begin{aligned} B_{i\bar{i}} &= 1 \qquad \text{or} \qquad -q \\ B_{\bar{i}i} &= q \qquad \text{or} \qquad -1. \end{aligned} \tag{29}$$

We will chose $B_{i\bar{i}} = 1$, $B_{\bar{i}i} = q$. The overall minus sign does not matter in our considerations. Including the nilpotency of the Grassmann sources j and ∂, we obtained $4n$ constraints on B.

(ii) The commutator relation (23b), which is valid for $|k - i| \geqslant 2$, can be calculated along the same lines as in (28). This results in

$$\begin{aligned} b_i b_k - b_k b_i = {} & (B_{\bar{i}k} B_{i\bar{k}} - B_{\bar{k}i} B_{k\bar{i}} - B_{\bar{i}\,\bar{k}} B_{ik} + B_{\bar{k}\,\bar{i}} B_{ki}) \mathbb{1} \\ & + (B_{\bar{i}k} + B_{k\bar{i}}) j_i \wedge \partial_k - (B_{i\bar{k}} + B_{k\bar{i}}) j_k \wedge \partial_i \\ & - (B_{\bar{i}\,\bar{k}} + B_{\bar{k}\,\bar{i}}) j_i \wedge j_k - (B_{ik} + B_{ki}) \partial_i \wedge \partial_k \\ \stackrel{!}{=} {} & 0. \end{aligned} \tag{30}$$

From this we obtain

$$B_{ik} = -B_{ki} \qquad B_{\bar{i}\,\bar{k}} = -B_{\bar{k}\,\bar{i}} \qquad B_{i\bar{k}} = -B_{\bar{k}i} \tag{31}$$

if $|i - k| \geqslant 2$. This leads to $3n(n-2)/2$ constraints on B.

(iii) The third relation is somewhat lengthy and yields

$$\begin{aligned} b_i b_{i+1} b_i - b_{i+1} b_i b_{i+1} = {} & (1+q)(B_{i\,i+1} B_{\overline{i+1}\,\bar{i}} - B_{i+1\,i} B_{\bar{i}\,\overline{i+1}}) \mathbb{1} \\ & + ((B_{i+1\,\bar{i}} + B_{\bar{i}\,i+1})(B_{i\,\overline{i+1}} + B_{\overline{i+1}\,i}) \\ & - (B_{i\,i+1} + B_{i+1\,i})(B_{\overline{i+1}\,\bar{i}} + B_{\bar{i}\,\overline{i+1}}) - q) j_i \wedge \partial_i \\ & + ((B_{\bar{i}\,\overline{i+1}} + B_{\overline{i+1}\,\bar{i}})(B_{i+1\,i} + B_{i\,i+1}) \\ & - (B_{\overline{i+1}\,i} + B_{i\,\overline{i+1}})(B_{i+1\,\bar{i}} + B_{\bar{i}\,i+1}) + q) j_{i+1} \wedge \partial_{i+1} \\ & - (1+q)(B_{i\,i+1} + B_{i+1\,i}) \partial_i \wedge \partial_{i+1} + (1+q)(B_{\bar{i}\,\overline{i+1}} + B_{\overline{i+1}\,\bar{i}}) j_i \wedge j_{i+1} \\ \stackrel{!}{=} {} & 0. \end{aligned} \tag{32}$$

This leads to

$$B_{i+1\,i} = -B_{i\,i+1} \qquad B_{\bar{i}\,\overline{i+1}} = -B_{\overline{i+1}\,\bar{i}} \qquad B_{\bar{i}\,i+1} = 1 - B_{i+1\,\bar{i}} \qquad B_{\overline{i+1}\,i} = q - B_{i\,\overline{i+1}} \tag{33}$$

which are $4(n-1)$ further constraints on B. All in all, we have to impose the constraints given in (26), (29), (31) and (33) on the bilinear form B of $4n^2$ arbitrary $\mathbb{K}$-valued parameters. We obtain

$$\text{No of constraints} = \frac{3n^2 + 13n - 8}{2} \tag{34}$$

and remain with

$$\text{No of degrees of freedom} = \frac{5n^2 - 13n + 8}{2}. \tag{35}$$

The explicit form of B can be derived from the constraints to be of the form (25). Since we remain with superfluous degrees of freedom, which might be set to arbitrary numbers, we have derived a whole set of Hecke algebra representations in $C\ell(V, B)$.

□

3.2. *On the structure of the algebra morphism ρ*

Since we have shown that we can find an image set of generators, we have to ask if they are free. In other words, we have to show whether or not the representation ρ is injective or equivalently has a trivial kernel. The mere calculation of the relations does not show this [41].

3.2.1. Some general aspects of the morphism. The main argument for a non-trivial kernel relies on dimensional considerations and belongs to the referee. We can calculate the maximal possible dimension spanned by the Clifford generators j and ∂ in the following way. We have $\dim V = \dim V^{\mathrm{T}} = n$ and thus $\dim \boldsymbol{V} = \dim V \oplus V^{\mathrm{T}} = 2n$. The corresponding Clifford algebra has 2^{2n} dimensions; however, since we are interested only in even-graded elements, we remain with $\dim C\ell(\boldsymbol{V}, B)_{\mathrm{even}} = 2^{2n-1} = 4^n/2$. On the other hand, it is known, for example [42, 43], that the dimension of $H_{\mathbb{K}}(n+1, q)$ is equivalent to S_{n+1}, iff q is generic, i.e. not a root of unity. One knows that this leads to $\dim H_{\mathbb{K}}(n+1, q) = \dim S_{n+1} = (n+1)!$. From table 1, we see that for $n \geqslant 5$ the number of linear independent algebra elements of $H_{\mathbb{K}}(n+1, q)$ exceeds the number of all independent even-dimensional Clifford elements.

Table 1. Dimensions of Clifford algebras, Hecke algebras and the kernel. For a special setting of the remaining freedoms more decisive results are given in section 3.2.2.

n	$\dim C\ell_{\mathrm{even}} = 4^n/2$	$\dim H_{\mathbb{K}}(n+1, q) = (n+1)!$	$\ker \rho$
1	2	2	0 (trivial)
2	8	6	0 (non-trivial)
3	32	24	?
4	128	120	?
5	521	720	$\neq 0$

It is clear that from the results in table 1 that we have a non-trivial kernel for generic q and $n \geqslant 5$.

Proof. Proof of the injectivity for $n = 2$. From the representation theory of the symmetric group, it is known [44] that the transposition class-sum uniquely characterizes all irreducible representations up to S_6. Furthermore, it was shown that in the Hecke case the corresponding construction is [45]

$$C_{n+1} := t_1 + t_2 + \cdots + t_n + \frac{1}{q}\sum_{i=1}^{n-1} t_i t_{i+1} t_i + \frac{1}{q^2}\sum_{i=1}^{n-2} t_i t_{i+1} t_{i+2} t_{i+1} t_i + \cdots + \frac{1}{q^{n-1}} t_1 \ldots t_n \ldots t_1. \tag{36}$$

This central operator now uniquely describes all irreducible representations of $H_{\mathbb{K}}(n+1, q)$. Different eigenvalues distinguish orthogonal representations, where multiple roots count the number of equivalent representations. In our case, we have

$$C_3 = b_1 + b_2 + \frac{1}{q} b_1 b_2 b_1 \tag{37}$$

where we calculate C_3 *in the* $C\ell$*-image*! The eigenvalues are

$$\left[(-q^2 - 2q)^1, \left(2 + \frac{1}{q}\right)^1, (1-q)^4 \right] \tag{38}$$

$$[-3, 3, (0)^4]_{q \to 1} \tag{39}$$

in agreement with [45], where the exponent gives the degeneracy. The fundamental representation is the orthogonal sum of the corresponding eigenspaces and is therefore of dimension $6 = 3! = \dim H_{\mathbb{K}}(3, q)$.

Remark. The elegant methods for studying Hecke algebras used in [45, 46] are not applicable, since they remain within the Hecke algebra setting and do not show the possible degeneracy of the image generators.

Remark. We might circumvent the dimensional argument against an injective homomorphism in the following way. Assume that we find a map $\rho^{(2r)} : H_{\mathbb{K}}(n+1, q) \to C\ell(\mathbb{K}^{2rn}, B)$ of higher degree with $\rho^{(2r)}(t_i) = b_i^{(2r)}$ so that the Hecke relations (23) are satisfied, $r = 1$ was our above case. If one then seeks n linear independent diagonal $2r$-elements $b_i^{(2r)}$, this requires $\boldsymbol{V}$ to be of dimension $r(2n)$, which leads to a total dimension of $2^{2nr}/2^r = 2^{(2n-1)r}$ for $C\ell^{+\cdots+}(\boldsymbol{V}, B) - r$ times the even part. Since r is in principle arbitrary, we may construct algebra morphisms $\rho^{(2r)}$ which do not fail to be injective for sufficiently large r *due to the dimensional argument*. However, this is not a proof that $\ker \rho^{(2r)} = 0$. There might be a non-trivial kernel for other reasons, as can be expected from the discussion in the next subsection.

Remark. If one looks at the $b_i^{(2r)}$ as composed objects, only low-dimensional cases are of physical interest.

Remark. A genuine account of q-symmetry and its relation to classical groups can be found in [7, 45]. From this work, it is clear that not all $H_{\mathbb{K}}(n+1, q)$ representations can occur if the invariance under the inner *bi-vector* product is also required. This can also be seen for symmetric groups [47]. A direct construction of spinor representations etc within the composite Clifford algebraic framework has intriguing details and will be given elsewhere, but see also [48].

3.2.2. Structure theorem for $\ker \rho$ *in the balanced situation.* In this section we will consider the properties of $\ker(\rho)$. This kernel has an intriguing structure, but we will be able to prove in relevant cases which structure governs the split into image and kernel.

Lemma 11. *The kernel of* ρ *strongly depends on the values, zero or not, of the chosen bilinear form* B*. Since the symmetric part is fixed by (23) up to special coordinate transformations leaving (23) invariant, the kernel dimension actually depends on the antisymmetric part of* B*.*

Proof. Calculate the Clifford product of two generators b_i, b_j:

$$b_i b_j = B_{\bar{i}j} B_{i\bar{j}} - B_{\bar{i}\bar{j}} B_{ij} + B_{\bar{i}j} j_i \wedge \partial_j - B_{i\bar{j}} j_j \wedge \partial_i - B_{\bar{i}\bar{j}} j_i \wedge j_j - B_{ij} \partial_i \wedge \partial_j + j_i \wedge j_j \wedge \partial_j \wedge \partial_i. \tag{40}$$

Examining the terms with two basis elements, we notice that they possess a pre-factor B_{ij} with or without barred indices. If such a factor is zero, the whole term vanishes. Furthermore, we cannot generate this Clifford basis monomial by another product of the b_i generators. □

Obviously this means, that the kernel dimension increases with every vanishing element in B. Since we had some freedom to choose the antisymmetric part of B, we can have larger and smaller kernels.

To reach our final goal, the description of the kernel of ρ for a special choice of parameters, we have to construct all Clifford basis monomials which can be reached by multiplying b_i generators. What is needed is an algebraic basis of the Hecke algebra $\{b_I\}$ to perform the map $\rho : H_{\mathbb{K}}(n+1, q) \mapsto C\ell_{n,n}$ on this basis, i.e. $\rho : \{b_I\} \mapsto \{j_{I'} \wedge \partial_{I''}\}$, where capitals denote index sets. We define some properties of an algebraic basis of the Hecke algebra [46].

Definition 12. *A minimal word $g_{t,s}$ is a sequence of b_i generators with step-wise decreasing indices $g_{t,s} = b_t b_{t-1} b_{t-2} \ldots b_{t-s}$.*

A minimal word has length $s + 1$. By examination we see that there are $(n^2 + n + 2)/2$ such words.

Definition 13. *A reduced word $r_{t_1,s_1,t_2,s_2,\ldots,t_m,s_m}$ is a product of minimal words g_{t_i,s_i}, where $s_i < t_{i+1}$ holds.*

Lemma 14. *The reduced words build up a basis of the Hecke algebra $H_{\mathbb{K}}(n+1, q)$ for generic q.*

Proof. We have built up the sets of minimal words $\{\mathbb{1}\}, \{\mathbb{1}, b_1\}, \ldots, \{\mathbb{1}, b_j, b_j b_{j-1}, \ldots, b_j b_{j-1} \ldots b_1\}, \ldots, \{\mathbb{1}, \ldots, b_n \ldots b_1\}$. Multiplying the terms element-wise and collecting them in a new set gives a totality of $1 * \cdots * n = n!$ different reduced words, which therefore span $H_{\mathbb{K}}(n+1, q)$. □

This basis provides us with the so-called regular representation, which contains all irreducible representations according to their multiplicity. It is convenient to introduce (q-) Young diagrams to label these representations and numberings, and hence Young tableaux to distinguish the different but isomorphic copies of the same type. We write $[n_1, \ldots, n_s]$ for a partition of $N = \sum n_i$ into s parts (Young diagram). Furthermore, we define $[1^r] = [1, \ldots, 1]$ r-times and $[0]$ as $[1^0]$ are defined as no-box, i.e. $[r, 0] \equiv [r]$ and $[r, 1^0] \equiv [r]$, to simplify special cases in our formulae. If we write m_i for the multiplicity and Y_i for the Young diagram, we obtain the decomposition formula

$$\oplus_I b_I = \oplus_i \oplus_{m_i} Y_i. \tag{41}$$

Remark. We loosely speak about Young diagrams and Young tableaux instead of q-Young diagrams and q-Young tableaux, since we only need some of their very general properties. In fact, q-Young diagrams are identical to Young diagrams, but q-Young tableaux should be handled with care. The q-Young operators corresponding to such tableaux show up a quite different structure, as the box content also becomes a function of q, see [46, 48].

We now apply the map ρ onto this basis to obtain the corresponding Clifford monomials and obtain as the lack of elements the dimension and structure of the kernel. To be able to do this, we specialize the bilinear form B in a suitable manner. From physical applications and driven by simplicity, we are interested in balanced multivectors only. Furthermore, we will introduce a new basis respectively a new more suitable bilinear form also denoted by B.

Definition 15. *A balanced multivector is a multivector $M_{I,\bar{J}}$ where the cardinality of the index sets of I and J are equal, i.e. $\#I = \#J$ (equal number of j and ∂ generators).*

It is trivial to see that the b_i generators are balanced; however, from (40) we see, that their products contain non-balanced multivectors. If we now set the doubly barred and doubly unbarred elements in the bilinear form B zero, which can be done since they all belong to the antisymmetric part of B, the product formula (40) goes over into

$$b_i b_j = B_{\bar{i}j} B_{i\bar{j}} + B_{\bar{i}j} j_i \wedge \partial_j - B_{i\bar{j}} j_j \wedge \partial_i + j_i \wedge j_j \wedge \partial_j \wedge \partial_i. \tag{42}$$

To simplify the bilinear form further, obtaining a bilinear form which is non-zero in every block off-diagonal entry, we define

$$\begin{aligned}
[B_{ij}] &:= \begin{bmatrix} 0 & B^u \\ B^d & 0 \end{bmatrix} \\
B^u &:= \begin{bmatrix}
q & -\frac{\lambda^2+q}{\lambda} & \nu & \dots & \dots & \nu \\
-\frac{\lambda^2+q}{\lambda} & q & -\frac{\lambda^2+q}{\lambda} & \nu & \dots & \nu \\
\nu & -\frac{\lambda^2+q}{\lambda} & q & -\frac{\lambda^2+q}{\lambda} & \ddots & \nu \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\nu & \dots & \nu & -\frac{\lambda^2+q}{\lambda} & q & -\frac{\lambda^2+q}{\lambda} \\
\nu & \dots & \dots & \nu & -\frac{\lambda^2+q}{\lambda} & q
\end{bmatrix} \\
B^d &:= \begin{bmatrix}
1 & \lambda & -\nu & \dots & \dots & -\nu \\
\frac{q}{\lambda} & 1 & \lambda & -\nu & \dots & -\nu \\
-\nu & \frac{q}{\lambda} & 1 & \lambda & \ddots & -\nu \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
-\nu & \dots & -\nu & \frac{q}{\lambda} & 1 & \lambda \\
-\nu & \dots & \dots & -\nu & \frac{q}{\lambda} & 1
\end{bmatrix} \\
&\nu \neq 0 \neq \lambda.
\end{aligned} \tag{43}$$

It is no restriction to our question to identify all free parameters as belonging to two species, since only their vanishing or not vanishing affects the structure of the kernel. In general, however, different parameter values lead to different results. The parameters ν and λ are distinct due to the fact that ν might be set to zero, while if λ is set to zero ρ is no longer a morphism from $H_{\mathbb{K}}(n+1, q)$ into $C\ell^+_{n,n}$. We may further notice that the symmetric part of the above bilinear form after performing the limit $q = \lambda \to 1$ and $\nu \to 0$ becomes one half the Cartan matrix $\boldsymbol{A}_n$ in each block G^u, G^d. The doubled symmetric part $G^u(q) = 2 * 1/2 * (B^u + B^{dT})$ can be defined as q-Cartan matrix of $\boldsymbol{A}_n(q)$. However, without the antisymmetric part of B this will not lead to a 'quantum' structure.

We are now prepared to give a structure theorem of $\ker\rho$ in this special situation.

Theorem 16 (Balanced morphism). *Let ρ be a Hecke algebra morphism as described in theorem 10 wrt the bilinear form given in (43). The morphism ρ maps all representations which have L-shaped Young diagrams $[(d-r), 1^r]$, one row one column, bijectively into the Clifford algebra $C\ell^+_{n,n}$. The kernel $\ker\rho$ consists of all other representations of $H_{\mathbb{K}}(n+1, q)$. The dimension of the image, $\mathrm{img}\,\rho = \rho(H_{\mathbb{K}}(n+1, q))$ is $\#\mathrm{img}\,\rho = (2n)!/(n!)^2$ while the dimension of the kernel is $\#\ker\rho = n! - \#\mathrm{img}\,\rho$. The representation spaces corresponding to different Young diagrams (including multiplicities) are given by the spaces of multivectors of different grades (Grassmannians), denoted by $G_{r,r}$.*

To prove theorem 16 we need some further results.

Lemma 17. *The Clifford reversion, denoted by $\tilde{\ }$, maps the q-symmetrizer and q-antisymmetrizer of the same symbols onto one another. That is, $\tilde{\ }$ interchanges rows and columns in Young diagrams and tableaux.*

Proof. Observe that the (anti)symmetrizer$(-/+)$ of two elements is given as

$$P_i^+ := \frac{q+b_i}{q+1} \qquad P_i^- := \frac{1-b_i}{q+1}. \tag{44}$$

We have

$$\begin{aligned} P_i^{+\sim} &= \frac{q}{q+1} + \frac{b_i{}^{\sim}}{q+1} \\ &= \frac{q+(1-q)-b_i}{q+1} \\ &= \frac{1-b_i}{q+1} = P_i^- \end{aligned} \tag{45}$$

and *vice versa*, since $\tilde{\tilde{\ }} = \mathrm{Id}$. This property can be enlarged to so-called Garnir elements [46] which are needed to construct the q-(anti)symmetrizer and, due to the fact that $\tilde{\ }$ is an anti-algebra morphism wrt the Clifford product, to the whole Young operators, which are products of row symmetrizers and column antisymmetrizers. □

Lemma 18. *If and only if a representation with a corresponding Young diagram is mapped into $C\ell^+_{n,n}$, then the tilted representation occurs there also. Tilted means representations for a Young diagram with rows and columns interchanged.*

Proof. We can expand the Clifford picture of the corresponding Young operator of the Young diagram into a Clifford product and sum of Clifford basis monomials. Then the Clifford reversion yields the tilted Young operator with lemma 17. □

We need some more information on the dimensionality of L-shaped diagrams.

Lemma 19. *The dimension of the representation of an L-shaped Young diagram of type $[(d-r), 1^r]$ is equal to the binomial coefficient $\binom{d}{r}$. The dimension of all such L-shaped representations, multiplicities included, is*

$$\sum_{r=0}^{d} \#[(d-r), 1^r] = \sum_{r=0}^{d} \binom{d}{r}^2 = \frac{(2d)!}{(d!)^2}. \tag{46}$$

The same conventions on $[0]$ and $[1^0]$ are assumed as above.

For a proof we refer to [47, 49]. In fact, only the 'hook-rule' and multiplicities are needed.

We can now prove theorem 16.

Proof of Theorem 16. We have seen in (42) that two generators when multiplied yield many Clifford monomials. We have to distinguish two types of them. The first is that of correlated indices, hence for example elements of type $I = \{ij\}$

$$e_{I\bar{I}} = e_{ij\bar{i}\,\bar{j}} = j_i \wedge j_j \wedge \partial_i \wedge \partial_j \tag{47}$$

which can easily be generated by multiplying the generators and picking out the highest grade. These might also be obtained by 'wedging' the generators.

The second type of Clifford monomials is built up from elements with two maximal distinct index sets $I \cap J = \emptyset$, for example $I = \{ij\}$, $J = \{rs\}$

$$e_{I\bar{J}} = e_{ij\bar{r}\,\bar{s}} = j_i \wedge j_j \wedge \partial_r \wedge \partial_s. \tag{48}$$

In (42) we found that two generators b_i, b_j produce such terms as

$$B_{i\bar{j}}\, j_j \wedge \partial_i \qquad B_{\bar{i}j}\, j_i \wedge \partial_j. \tag{49}$$

These terms are non-zero if and only if $B_{i\bar{j}}$ and $B_{\bar{i}j}$ are non-zero. To be able to generate all possible elements of this second form, we have to require that $B_{i\bar{j}} \neq 0 \neq B_{\bar{i}j}$ $\forall\, i, j$ as is the case in (43). Furthermore, one has to notice that the maximal cardinality of I and J is $n/2$, since $I \cup J = \{1, \ldots, n\}$ and $I \cap J = \emptyset$. This means that only n generators are needed to produce these elements. All other elements required are mixed forms of these two types. The length of the reduced words has a range from 0 to $n!$ and is thus rich enough to yield every balanced monomial, as can be seen by iteration of the above arguments. This proves that the image of ρ consists of all balanced Clifford monomials in $C\ell^+_{n,n}$. In terms of a formula,

$$\operatorname{img} \rho(H_{\mathbb{K}}(n+1, q)) = \oplus_{i=0}^{n} G_{i,i} \tag{50}$$

where $G_{i,i}$ are Grassmannians of balanced multivector spaces.

To complete the proof, we have to give the dimensionalities and identifications with representations of these spaces. Obviously, $\#G_{0,0} = \#G_{n,n} = 1$, and the other spaces are built up as follows (r times r factors)

$$\begin{aligned} &G_{r,r} = V \wedge \cdots \wedge V \wedge V^{\mathrm{T}} \wedge \cdots \wedge V^{\mathrm{T}} \\ &\#V \wedge \cdots \wedge V = \binom{n}{r} \\ &\#V^{\mathrm{T}} \wedge \cdots \wedge V^{\mathrm{T}} = \binom{n}{r}. \end{aligned} \tag{51}$$

We obtain the following formula for the dimensions:

$$\#\operatorname{img} \rho = \#\oplus_{r=0}^{n} G_{r,r} = \sum_{r=0}^{n} \binom{n}{r}^2 = \frac{(2n)!}{(n!)^2}. \tag{52}$$

To complete the claims of theorem 16, we have to show that the spaces $G_{r,r}$ of balanced multivectors of definite grade provide representations according to the L-shaped Young diagrams of type $[(n-r), 1^r]$. From lemma 19 we know that the dimension formula

$$\#[(n-r), 1^r] = \#G_{r,r} \tag{53}$$

holds for all r, where 0 and n is included by our above definition. It remains to show that these spaces are invariant under the action of the generators b_i.

There are three cases. We can define a Lie action of the b_i generators on the $G_{r,r}$ spaces via a commutator $X_{r\bar{r}} \in G_{r,r}$

$$b_i \bullet X_{r\bar{r}} := [b_i, X_{r\bar{r}}]. \tag{54}$$

Since the b_i are bi-vector elements this action is grade preserving. A little bit surprising is the fact that we can also find a left (right) representation on the spaces $G_{r,r}$, defined as $Y_{r\bar{r}} \in G_{r,r}$,

$$b_i \bullet Y_{r\bar{r}} := b_i Y_{r\bar{r}} \in G_{r,r}. \tag{55}$$

The third possibility is given by the automorphism action. Calculating

$$b_i^{-1} = -\frac{1}{q}[(1-q) - b_i] = -\frac{1}{q} b_i\,\tilde{} \tag{56}$$

we have with $Z_{r\bar{r}} \in G_{r,r}$

$$b_i \bullet Z_{r\bar{r}} := b_i Z_{r\bar{r}} b_i^{-1}. \tag{57}$$

This map is grade preserving since the b_i generators are in the Clifford Lipschitz group, see [48], even if the reversion does not respect the grades!

Obviously, all types of representations are degenerate with the multiplicity m_i corresponding to the Young diagram Y_i. To get the irreducible representations one would have to pick out a special basis in $G_{r,r}$ which splits the Grassmannians into $H_{\mathbb{K}}(n+1, q)$ irreducible parts according to numberings Γ in Young tableaux $Y_{i,\Gamma}$

$$G_{r,r} = \oplus_{m_i} Y_i = \oplus_\Gamma Y_{i,\Gamma}. \tag{58}$$

The kernel of the balanced morphism ρ consists thus of all irreducible representations with non-L-shaped Young diagrams and is of dimension $\#\mathrm{ker}\,\rho = n! - \#\mathrm{img}\,\rho$. $\square$

Remark. The balanced case is closely related to supersymmetry, as can be seen from [46], where supersymmetric Schur functions are used to characterize the representations. Furthermore, it is known that supersymmetric, or better graded, groups gain representations which are characterized by L-shaped Young diagrams with p-rows and q-columns in the case of a $G(p|q)$ supergroup [50]. See as a reference and for further literature [51].

Remark. From our results, it is clear that $H_{\mathbb{K}}(3, q)$ $(n = 2)$ is the only case where we have an algebra isomorphism. The next higher dimension $H_{\mathbb{K}}(4, q)$ $(n = 3)$ has a non-L-shaped Young diagram of type $[2, 2]$. It is of dimension two and multiplicity two, which gives a four-dimensional kernel and a space of 20 dimensions for the balanced multivectors in the image, $(2*3)!/(3!)^2 = 20$, together: $4! = 24 = 20 + 4$. The basis sets and dimensions have been checked independently by computer algebra up to the case $H_{\mathbb{K}}(5, q)$ $(n = 4)$. There the kernel consists of the representations $[3, 2]$ and $[2, 2, 1]$, which are 50-dimensional including multiplicities, each being representation five-dimensional with multiplicity five $5*5+5*5 = 50$. That is again, using $(2*4)!/(4!)^2 = 70$ one has finally the decomposition $5! = 120 = 70 + 50$.

Remark. Since it was essential that all block-off-diagonal matrix elements of the bilinear form (43) had to be non-zero, this has an implication on physical systems underlying such a model. From [52] we know that this situation is obtained for mixed states only, i.e. states which can be decomposed into convex combinations of pure or vector states. The q-symmetric case is thus generically connected to non-Fock vacua and may describe quasi-particles, which would support our point of view that q-symmetry is a symmetry of objects with internal structure or say composites, without addressing this structure explicitly.

3.2.3. Further observations on related algebraic systems. Since we have proved theorem 10 for the quadratic Hecke condition (23a), we have to give some comments on the other choices of the quadratic or higher relations. One finds in the literature at least the following types of relations

$$b_i^2 = (1-q)b_i + q \qquad t_i^2 = (q-q^{-1})t_i + 1 \qquad e_i^2 = \tau e_i \qquad u_i^Q = 1. \tag{59}$$

In general, one has a quadratic—or higher order, see u_i—relation

$$x_i^2 = g(q)x_i + h(q) \tag{60}$$

where g and h are meromorphic functions of q. The question, if there is a transformation connecting the b_i's and in general the x_i's, addresses the number of equivalence classes of quadratic relations. We can reformulate the above equations into $Q(x) = E$ where E is a constant, and we have to classify quadratic forms. This can be done over $\mathbb{R}$ and $\mathbb{C}$ with the help of the Brauer group $B(\mathbb{K})$ [53, 36]. This group is trivial for $\mathbb{C}$ since the complex field is algebraically closed and isomorphic to $\{-1, +1\}$ as a multiplicative group in the case of $\mathbb{R}$

$$B(\mathbb{K}) \cong \frac{\mathbb{K}^*}{\mathbb{K}^2}. \tag{61}$$

However, this classification only takes (23a) into account. It is easy to calculate that a transformation of the type

$$x_i = a(q) + b(q)b_i \tag{62}$$

with meromorphic $a(q), b(q)$ does not alter (23b). However, (23c) is in general *not* invariant under such a transformation, which is well known in the literature. In fact, one should restrict the allowed transformations to those which leave (23) unchanged. However, interesting cases do not respect the third law in general. As an example, one arrives at the relations of a Temperley–Lieb algebra [54], where the third relation is given using $e_i = (q\mathbb{1} + b_i)/(q+1)$, $\tau = (2+q+q^{-1})^{-1}$, compare with (23c), as

$$e_ie_{i+1}e_i - \tau e_i = e_{i+1}e_ie_{i+1} - \tau e_{i+1} \qquad (=0). \tag{63}$$

Here $(=0)$ indicates that usually the left- and right-hand sides are each set to zero which is in fact a new relation. The transformation from the Hecke algebra into the Temperley–Lieb algebra would be of the above-described invariance set only if $\tau = 0$, which leads to $q \in \{0, \infty\}$. In our approach, such a relation can easily be obtained simply by another choice of the bilinear form B or of course by the above transformation. It remains, however, to find a connection between traces employed in the phenomenological models and states on our algebra. Such states were introduced in [52] and provide a very rigid structure on $C\ell(V, B)$. These states are necessary to be able to calculate invariants and physical outcomes and to be able to show (in)equivalence between different (re)presentations. This intriguing problem will be addressed elsewhere.

4. Conclusion

As our main tool we have used Clifford geometric algebras of multivectors, which provide a generalization of ordinary Clifford algebras of quadratic spaces to such a pair as (V, B). The bilinear form does not have any symmetry in general and is thus not bijectively connected to a quadratic form. Every bilinear form B with the same symmetric part G gives rise to the same Clifford algebra. Taking the full B into account allows one to endow the Clifford algebra with a unique $\mathbb{Z}_n$-grading. The Clifford algebra corresponding to B built over the $\mathbb{Z}_n$-graded space $\bigwedge(V)$ is called the Clifford algebra of multivectors [8].

We proved the theorem that, due to an appropriate choice of the bilinear form B in $C\ell(\mathbb{K}^{2n}, B)$, it is possible to find n bivectors b_i which generate the Hecke algebra $H_{\mathbb{K}}(n+1, q)$, which is not free at least for $n \geqslant 5$. The proof was straightforward. Since we achieved a large number of remaining freedoms in the Clifford bilinear form B, these parameters might be used to obtain spectral parameters in the braid relation, which then mutates into the Yang–Baxter equation. This will be considered elsewhere.

We succeeded in finding a decisive answer to the structure of the kernel of this morphism from $H_{\mathbb{K}}(n+1, q)$ into $C\ell^{+}_{n,n}$. The spaces of balanced multivectors played an important role. A connection between representation spaces and graded spaces was thereby established. A connection to supersymmetric Schur functions occurs.

In the theory of (quantum) Yang–Baxter R-matrices, it seems not to be clear which interpretation to the symmetries and formulae should be given, see [55]. This situation changes in our radical approach, which might be its greatest advantage. Since the Clifford algebra already has an interpretation in physical terms, we have to look at the b_i generators as composite entities. This supports our opinion stated in the introduction and also promoted in [27] that q-symmetry might be connected with composite effects. A decision of this conjecture requires further work, especially on the states involved in the calculation of invariants, see [52].

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